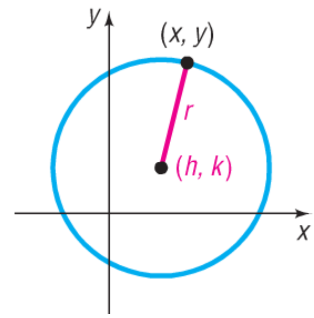
Circles

Circle*:** The set of all points in the *xy*-plane that are a fixed distance *r*, called the ***radius, from a fixed point (h, k), called the ***center*** of the circle.



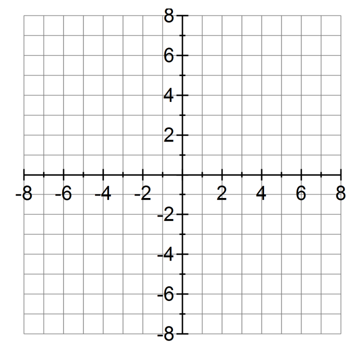
Standard Form of the Equation of a Circle with radius *r* and center (h, k):

$$(x-h)^2 + (y-k)^2 = r^2$$

***General Form of the Equation of a Circle*:**

$$x^2 + y^2 + ax + by + c = 0$$

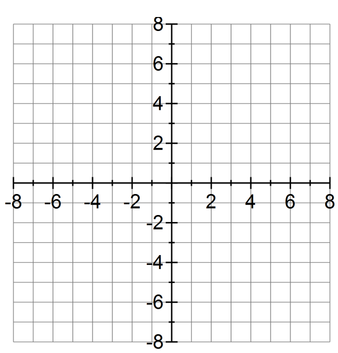
Example: Write the standard form of the equation and the general form of the equation of the circle with radius $r = 4$ and center $(h, k) = (4, -3)$. Then graph the circle.

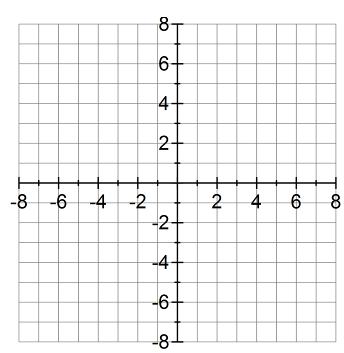


Examples: Find the center (h, k) and radius *r* of each circle, graph the circle, and find the intercepts, if any.

a) $(x+1)^2 + (y-2)^2 = 25$

b) $3(x+1)^2 + 3(y-1)^2 = 6$





★ To find the standard form of the equation of a circle when you know the general form, complete the square for both x and y.

Examples: Find the standard form of the equation of each circle. State the center and radius of the circle.

a) $x^2 + y^2 - 6x + 2y + 9 = 0$

b) $2x^2 + 2y^2 + 8x - 8 = 0$

***Distance Formula*:** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

***Midpoint Formula*:** $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

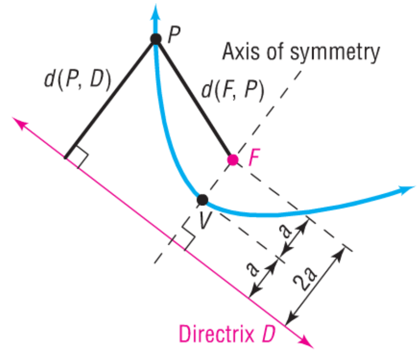
Examples: Find the standard form of the equation of each circle.

a) Center at $(1,0)$ and containing the point $(-3,2)$.

b) Endpoints of a diameter at $(4,3)$ and $(0,1)$.

The Parabola

Parabola**:** The collection of all points P in the plane that are the same distance from a fixed point F, called the ***focus*** of the parabola, as they are from a fixed line D, called the ***directrix*** of the parabola.



Axis of Symmetry**:** The line through the focus F and perpendicular to the directrix D.

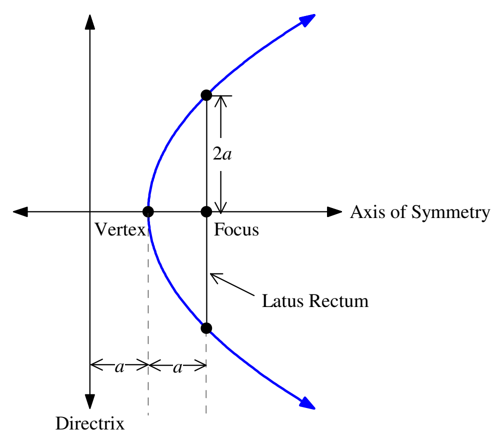
Vertex**:** The point of intersection of the parabola with its axis of symmetry.

General Forms of the Equation of a Parabola with Vertex (h,k)

a **= Distance from Focus to Vertex** ***a*** **= Distance from Vertex to Directrix**

Equation	Description	Picture
$(y-k)^2 = 4a(x-h)$	Opens Right, Axis of Symmetry parallel to *x*-axis	
$(y-k)^2 = -4a(x-h)$	Opens Left, Axis of Symmetry parallel to *x*-axis	
$(x-h)^2 = 4a(y-k)$	Opens Up, Axis of Symmetry parallel to *y*-axis	
$(x-h)^2 = -4a(y-k)$	Opens Down, Axis of Symmetry parallel to *y*-axis	

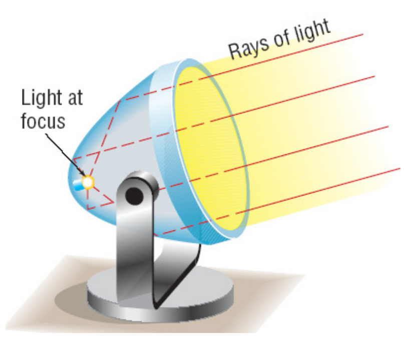
Latus Rectum**:** The line segment with endpoints on the parabola that passes through the focus and is perpendicular to the axis of symmetry. Each of the endpoints is at a distance of $2a$ from the focus.

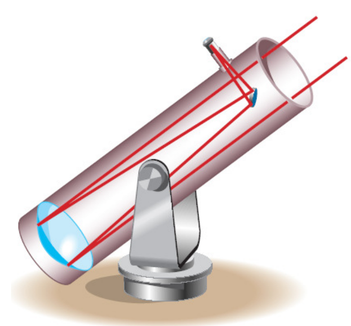


Paraboloid of Revolution**:** A surface formed by rotating a parabola about its axis of symmetry.

Suppose a mirror is shaped like a paraboloid of revolution. If a light (or other radiation source) is placed at the focus of the parabola, all the rays emanating from the light will reflect off the mirror in lines parallel to the axis of symmetry.

Conversely, when rays of light from a distant source strike the surface of a parabolic mirror, they are reflected to a single point at the focus. This fact is used in the design of telescopes and other optical devices.

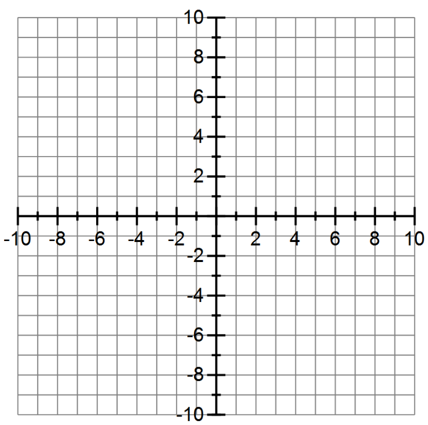


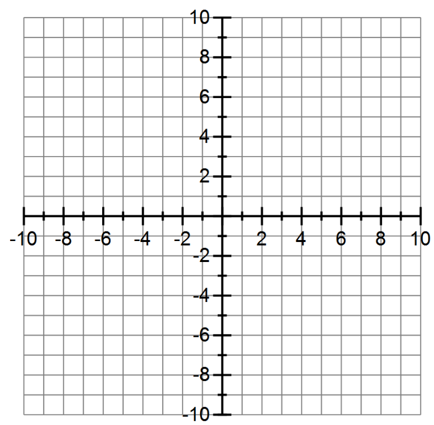


Examples: Graph the following parabolas. State the vertex, focus, axis of symmetry, directrix, length of latus rectum, and direction of opening.

a) $x^2 = 8y$

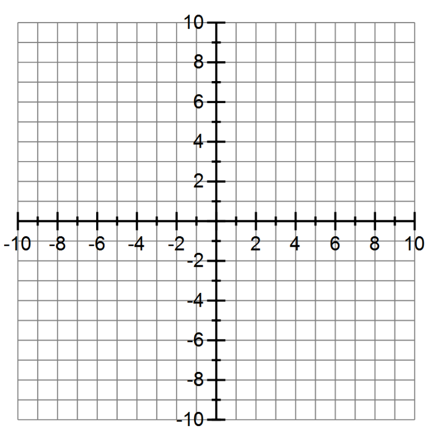
b) $y^2 = -16x$

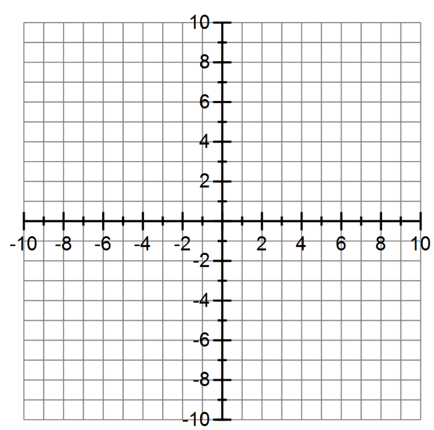




c) $(x-2)^2 = -12(y+1)$

d) $(y+3)^2 = 2(x-4)$





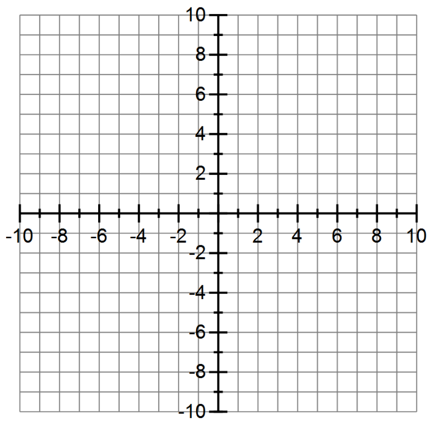
Examples: Write each equation in standard form.

a) $y = x^2 + 2x + 2$

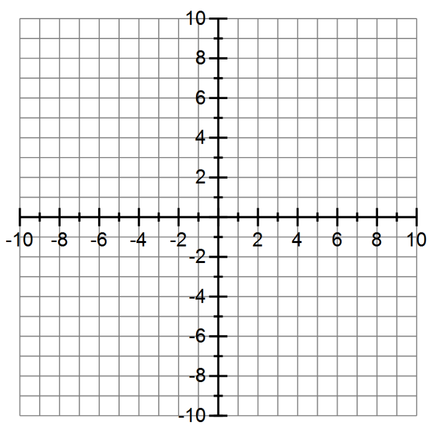
b) $x + y^2 = 6y - 3$

Examples: Find the equation of the parabola described. Find the two points that define the latus rectum and graph the equation.

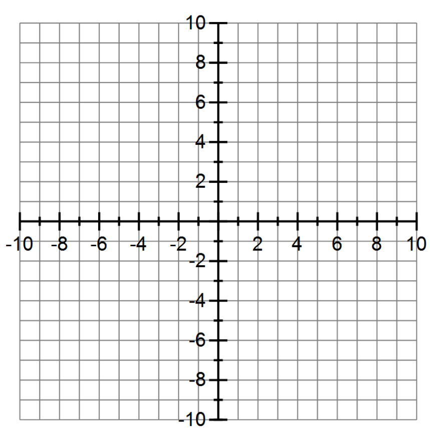
a) Vertex: $(0,0)$; Focus: $(0,-3)$



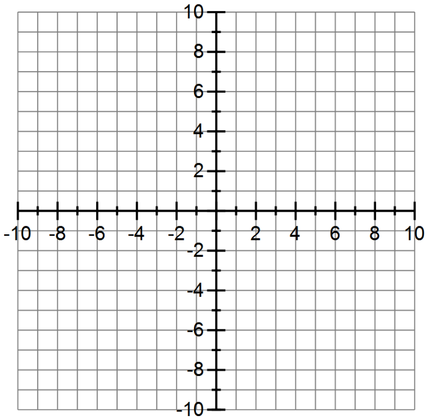
b) Vertex: $(3,-2)$; Focus: $(5,-2)$



c) Vertex: $(-1,4)$; Directrix: $x=1$

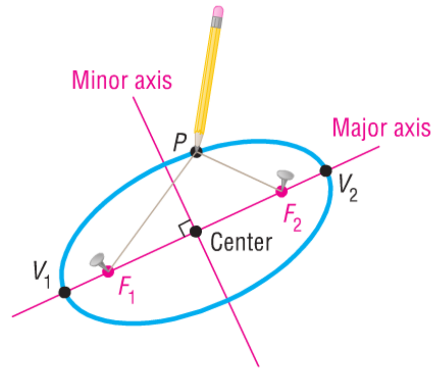


d) Vertex: $(0,-1)$; Axis of Symmetry: y-axis; Contains the point $(4,5)$



Example: A cable TV receiving dish is in the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 6 feet across its opening and 2 feet deep.

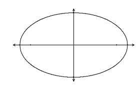
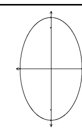
The Ellipse



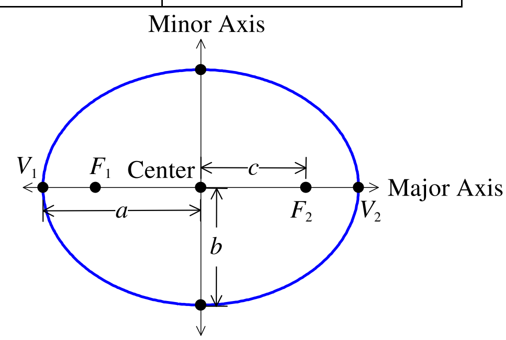
Ellipse: The collection of all points in the plane, the sum of whose distances from two fixed points, called the ***foci***, F_1 and F_2, is a constant.

Major Axis: The line containing the foci.
Center: The midpoint of the line segment joining the two foci.
Minor Axis: The line through the center and perpendicular to the major axis.
Vertices: The points of intersection of the ellipse and the major axis.
Covertices: The points of intersection of the ellipse and the minor axis.

Standard Form of the Equation of an Ellipse with Center at (h,k)

Equation	Description	Picture
$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ $a>b>0$ and $a^2-b^2=c^2$	Major axis parallel to *x*-axis	
$\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$ $a>b>0$ and $a^2-b^2=c^2$	Major axis parallel to *y*-axis	

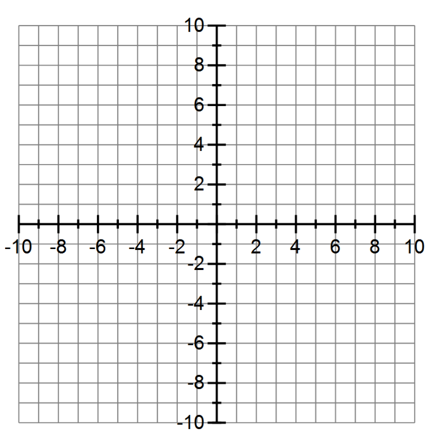
***a* = Distance from center to vertices**
***b* = Distance from center to covertices**
***c* = Distance from center to foci**

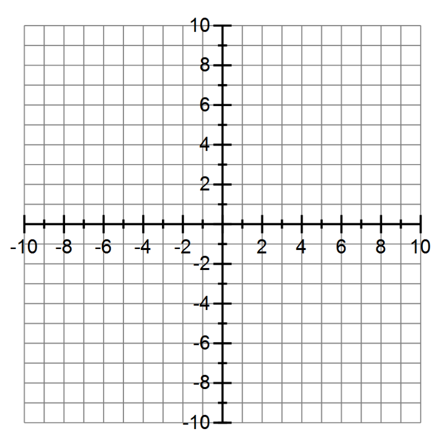


Examples: Find the center, foci, and vertices of each ellipse. Graph each equation.

a) $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

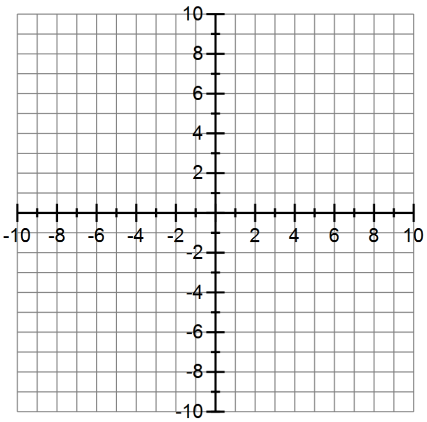
b) $\dfrac{x^2}{16}+\dfrac{y^2}{36}=1$

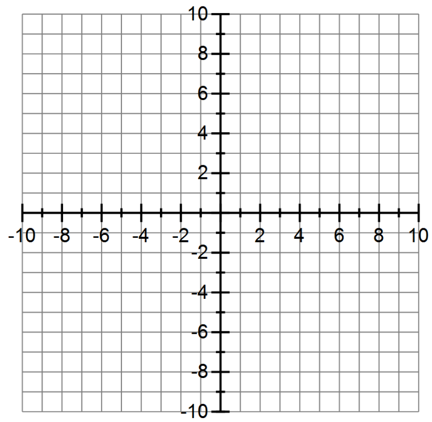




c) $\dfrac{(x+1)^2}{81}+\dfrac{(y-2)^2}{49}=1$

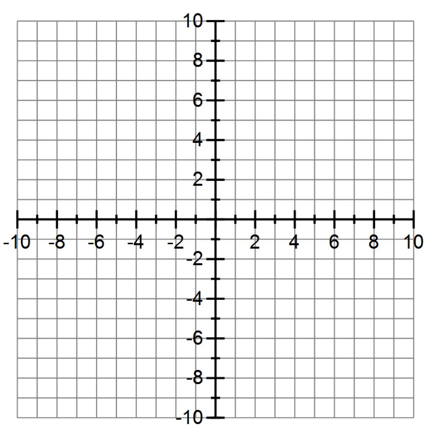
d) $9(x-3)^2+(y+2)^2=18$

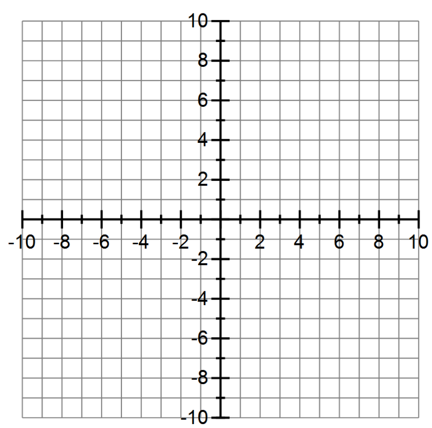




e) $x^2+9y^2+6x-18y+9=0$

f) $4x^2+y^2+4y=0$





Examples: Write the equation of the ellipse having the given characteristics.

a) Foci at $(1,2)$ and $(-3,2)$; Vertex at $(-4,2)$

b) Vertices at $(-1,5)$ and $(-1,-3)$; $c = 1$

c) Center at $(1,2)$; Focus at $(1,4)$; Contains $(2,2)$

Example: A hall 100 feet in length is to be designed as a whispering gallery. If the foci are located 25 feet from the center, how high will the ceiling be at the center?

Example: A bridge is to be built in the shape of a semielliptical arch and is to have a span of 100 feet. The height of the arch, at a distance of 40 feet from the center is to be 10 feet. Find the height of the arch at its center.

The Hyperbola

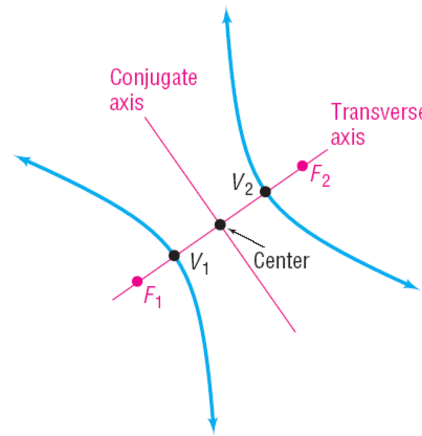
Hyperbola**:** The collection of all points in the plane, the difference of whose distances from two fixed points, called the **foci,** is a constant.

Transverse Axis**:** The line containing the foci.

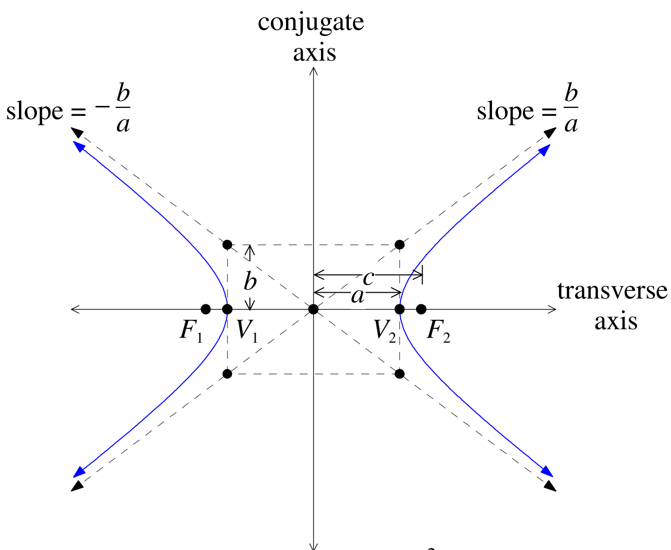
Center**:** The midpoint of the line segment joining the foci.

Conjugate Axis**:** The line through the center and perpendicular to the transverse axis.

Branches**:** The separate curves of the hyperbola. They are symmetric with respect to the transverse axis, conjugate axis, and center.

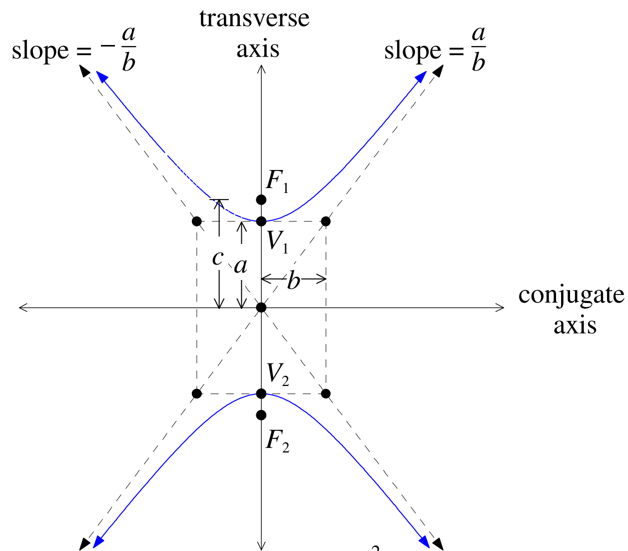


Vertices**:** The points of intersection of the hyperbola and the transverse axis.



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$a^2 + b^2 = c^2$$



$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$a^2 + b^2 = c^2$$

$a =$ distance from center to vertices

$c =$ distance from center to foci

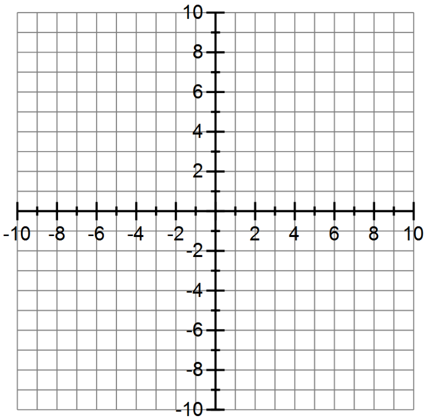
b used to find the width of branches and slope of asymptotes

When finding the equations of the asymptotes, remember that $m = \dfrac{\text{change in } y}{\text{change in } x}$, or, in this case,

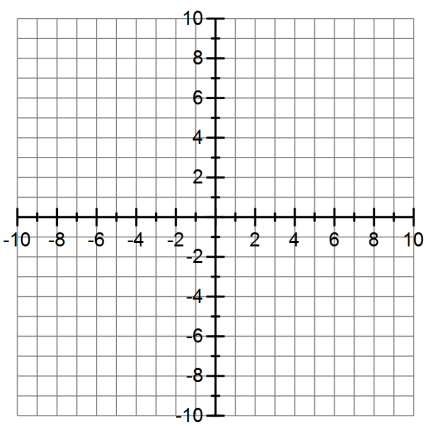
$m = \pm\dfrac{\sqrt{\#\text{ under } y^2 \text{ term}}}{\sqrt{\#\text{ under } x^2 \text{ term}}}$, then use point slope form $y - y_1 = m(x - x_1)$ with the center (h,k) as (x_1, y_1).

Examples: Find the center, transverse axis, vertices, foci, and asymptotes. Graph each equation.

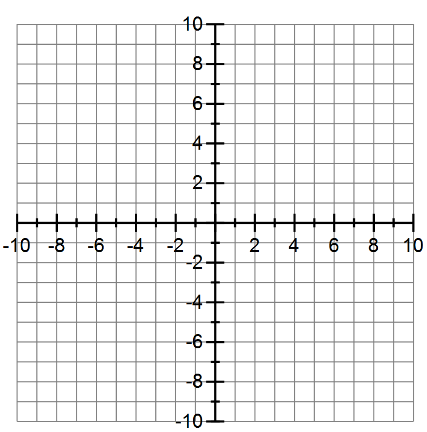
a) $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$



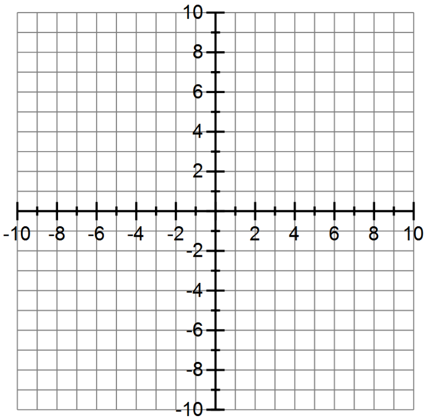
b) $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$



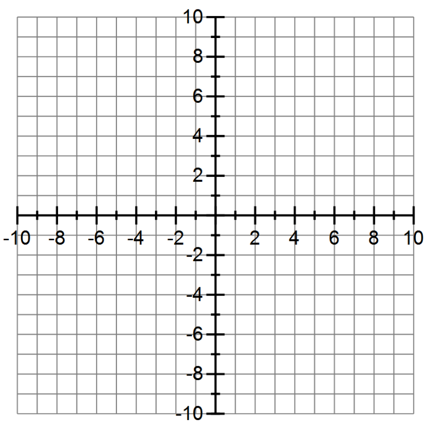
c) $4x^2 - 9y^2 = 36$



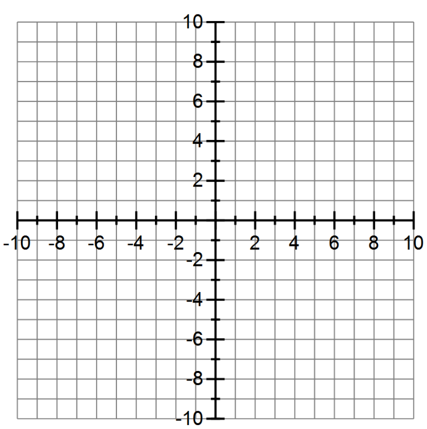
d) $\dfrac{(y+3)^2}{4}-\dfrac{(x-2)^2}{9}=1$



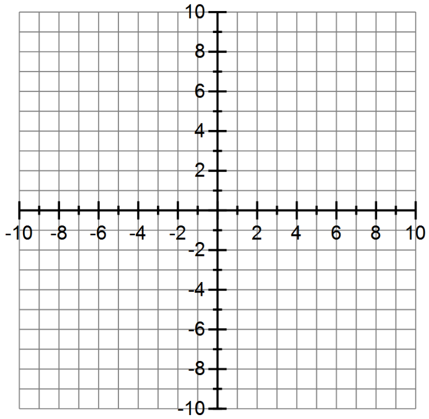
e) $(x+4)^2-(y-3)^2=9$



f) $y^2-x^2-4y+4x-1=0$



g) $2y^2 - x^2 + 2x + 8y + 3 = 0$



Examples: Write the equation of the hyperbola described.

a) Center at $(1,4)$; Focus at $(-2,4)$; Vertex at $(0,4)$

b) Focus at $(-4,0)$; Vertices at $(-4,4)$ and $(-4,2)$

Example: Suppose that two people standing 1 mile apart both see a flash of lightning. After a period of time, the person standing at point A hears the thunder. One second later, the person standing at point B hears the thunder. If the person at B is due west of the person at A and the lightning strike is known to occur due north of the person standing at point A, where did the lightning strike? The speed of sound is 1100 feet per second.

Systems of Linear Equations: Matrices

***Matrix*:** A rectangular array of numbers.
***Size of a Matrix*:** # of rows (m) $\times$ # of columns (n)

$$\begin{array}{c} \\ \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } i \\ \vdots \\ \text{row } m \end{array}\begin{array}{c} \begin{array}{ccccccc} \text{col } 1 & \text{col } 2 & \cdots & \text{col } j & \cdots & \text{col } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Example: Name the size of $\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 1 & -5 \end{bmatrix}$

Each number a_{ij} of a matrix is called an ***entry*.** Each entry has two indices: i is the ***row index***, which tells what row the entry is on, and j is the ***column index***, which tells what column the entry is in. For example, a_{23} refers to the entry in the 2nd row and 3rd column.

***Augmented Matrix*:** A matrix used to represent a system of equations.

If we do not include the constants to the right of the equal sign, that is, to the right of the bar in the augmented matrix, the resulting matrix is called the ***coefficient matrix*** of the system.

Examples: Write the augmented matrix and coefficient matrix for each system of equations.

a) $\begin{cases} 3x-4y=-6 \\ 2x-3y=-5 \end{cases}$

b) $\begin{cases} 4x-3y+2z=0 \\ 3x-z+2=0 \\ x+y-9=0 \end{cases}$

Examples: Write the system of equations corresponding to each augmented matrix.

a) $\left[\begin{array}{cc|c} 5 & -2 & 6 \\ 3 & 1 & -4 \end{array}\right]$

b) $\left[\begin{array}{ccc|c} 1 & -3 & 3 & -5 \\ -4 & -5 & -3 & -5 \\ -3 & -2 & 4 & 6 \end{array}\right]$

***Row Operations*:**

1. Switch the order of the rows.
2. Multiply or divide a row by a non-zero constant.
3. Add a multiple of one row to another row.

Examples: Apply the following row operations, in order, to the augmented matrix.

$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -5 & -4 \end{array}\right]$

a) $R_2 = -2r_1 + r_2$

b) $R_1 = 3r_2 + r_1$

Example: Find a row operation that will result in the matrix $\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]$ having a 0 in row 1, column 2.

***Row Echelon Form*:**
1. The first non-zero entry on each row is 1, and only zeros appear below it.
2. The first 1 in each row is to the right of the leading 1 in any row above.
3. Any rows that contain all 0's to the left of the vertical bar appear at the bottom.

***Reduced Row Echelon Form*:**
Entries are 0 above, as well as below, the leading 1's in each row.

***Consistent System*:** A system of equations with at least one solution.
***Inconsistent System*:** A system of equations with no solution.
***Dependent System*:** A system of equations with infinitely many solutions. (At least one of the variables *depends* on the values of the other variables.)

Examples: The reduced row echelon form of a system of equations is given. Write the system of equations corresponding to the given matrix. Use x, y, z or x_1, x_2, x_3, x_4 as variables. Determine whether the system is consistent or inconsistent. If it is consistent, give the solution.

a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

b) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$

c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$

Solving Systems of Equations Using Row Operations:
- Perform row operations to get a leading 1 in row 1, column 1.
- Once you have a leading 1, perform row operations to get zeros above and below it.
- Work to get a leading 1 in row 2, column 2, then get zeros above and below it.
- Keep going, working top to bottom and left to right.
- Be careful and show your work! It is much better to do it right the first time than to go back and search for a mistake.

Examples: Solve each system of equations using matrix row operations. If the system has no solution, say that it is inconsistent.

a) $\begin{cases} \frac{1}{2}x + y = -2 \\ x - 2y = 8 \end{cases}$

b) $\begin{cases} 2x - 3y - z = 0 \\ -x + 2y + z = 5 \\ 3x - 4y - z = 1 \end{cases}$

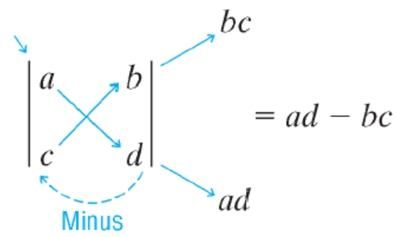
c) $\begin{cases} x + y + z + w = 4 \\ 2x - y + z = 0 \\ 3x + 2y + z - w = 6 \\ x - 2y - 2z + 2w = -1 \end{cases}$

Example: A doctor's prescription calls for a daily intake of a supplement containing 40 mg of vitamin C and 30 mg of vitamin D. Your pharmacy stocks three supplements that can be used: one contains 20% vitamin C and 30% vitamin D; a second, 40% vitamin C and 20% vitamin D; and a third, 30% vitamin C and 50% vitamin D. Create a table showing the possible combinations that could be used to fill the prescription.

Determinants

If a, b, c, and d are four real numbers, the symbol $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a ***2 by 2 determinant.*** Its value is the number $ad - bc$; that is,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$



Examples: Evaluate each 2 by 2 determinant.

a) $\begin{vmatrix} 8 & -3 \\ 4 & 2 \end{vmatrix} =$

b) $\begin{vmatrix} 6 & 1 \\ 5 & 2 \end{vmatrix} =$

c) $\begin{vmatrix} 2 & -3 \\ -7 & -5 \end{vmatrix} =$

d) $\begin{vmatrix} -9 & -7 \\ 1 & -6 \end{vmatrix} =$

3 by 3 Determinants: Expansion by Minors

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The 2 by 2 determinants shown above are called ***minors*** of the 3 by 3 determinant. The ***minor*** $\boldsymbol{M_{ij}}$ is the determinant that results from removing the ith row and jth column of the determinant.

The signs in the equation for a 3 by 3 determinant alternate. To determine whether to add or subtract each term, consider the expression $(-1)^{i+j}$, where $i + j$ is the sum of the row and column number of the entry associated with each minor. If $i + j$ is even, $(-1)^{i+j} = 1$, so we add, and if $i + j$ is odd, $(-1)^{i+j} = -1$, so we subtract.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

+ means "same sign"
– means "opposite sign"

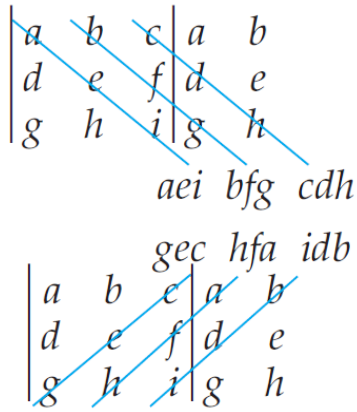
The process used to find the value of a 3 by 3 determinant is called ***expanding across a row or column.*** Select one row or column, and multiply each entry in that row or column by its minor. Use the rules above to decide whether to add or subtract. The value of the determinant is the same no matter which row or column you use.

Examples: Evaluate each 3 by 3 determinant using expansion by minors.

a) $\begin{vmatrix} 1 & 3 & -2 \\ 6 & 1 & -5 \\ 8 & 2 & 3 \end{vmatrix} =$

b) $\begin{vmatrix} -2 & 1 & 0 \\ 1 & 2 & 3 \\ -1 & 4 & 2 \end{vmatrix} =$

3 by 3 Determinants: Using Diagonals



1. Rewrite the first two columns to the right of the determinant.
2. Draw diagonals from each element in the top row downward to the right. Find the product of the entries on each diagonal.
3. Draw diagonals from each element in the bottom row upward to the right. Find the product of the entries on each diagonal.
4. Add the products of the first set of diagonals and subtract the products of the second set of diagonals.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (gec + hfa + idb)$$

Example: Evaluate each discriminant using diagonals.

a) $\begin{vmatrix} 4 & -1 & 2 \\ 6 & -1 & 0 \\ 1 & -3 & 4 \end{vmatrix}$

b) $\begin{vmatrix} 7 & -1 & 3 \\ -4 & 2 & 2 \\ 0 & 1 & -3 \end{vmatrix}$

Cramer's Rule for a 2 by 2 System of Equations

The solution to the system of equations $\begin{cases} a_{11}x + a_{12}y = c_1 \\ a_{21}x + a_{22}y = c_2 \end{cases}$ or $\left[\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \end{array}\right]$ is given by:

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \text{ and } y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{12} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \text{ provided that } D \neq 0.$$

D is the determinant of the coefficient matrix, D_x is formed by replacing the entries in the x-column of D by the constants on the right-hand side of the equation, and D_y is formed by replacing the entries in the y-column of D by the constants on the right-hand side of the equation.

Examples: Solve by using Cramer's Rule.

a) $\begin{cases} 5x - y = 13 \\ 2x + 3y = 12 \end{cases}$

b) $\begin{cases} 2x + 4y = 16 \\ 3x - 5y = -9 \end{cases}$

Cramer's Rule for a 3 by 3 System of Equations

The solution to the system of equations $\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$ or $\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$ is given by:

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D},\ \text{provided } D \neq 0,\ \text{where}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},\ D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix},\ D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix},\ \text{and } D_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}.$$

Examples: Solve by using Cramer's Rule.

a) $\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$

b) $\begin{cases} x + 4y - 3z = -8 \\ 3x - y + 3z = 12 \\ x + y + 6z = 1 \end{cases}$

Properties of Determinants:

1. Interchanging any two rows or columns of a matrix changes the sign of the determinant.
2. If all the entries in any row or column of a matrix equal zero, the value of the determinant is zero.
3. If any two rows or columns of a matrix have corresponding entries that are equal, the value of the determinant is zero.
4. If any row or column of a determinant is multiplied by a nonzero number *k*, the value of the determinant is also changed by a factor of *k*.
5. If the entries of any row or column of a matrix are multiplied by a nonzero number *k* and the result is added to the corresponding entries of another row or column, the value of the determinant remains unchanged.

Examples: Use the properties of determinants to find the value of each determinant if it is known that

$$\begin{vmatrix} a & b & c \\ 5 & 3 & 1 \\ p & q & r \end{vmatrix} = 7.$$

a) $\begin{vmatrix} c & b & a \\ 1 & 3 & 5 \\ r & q & p \end{vmatrix}$

b) $\begin{vmatrix} p & q & r \\ 10 & 6 & 2 \\ a & b & c \end{vmatrix}$

c) $\begin{vmatrix} a-p & b-q & c-r \\ 5 & 3 & 1 \\ p & q & r \end{vmatrix}$

d) $\begin{vmatrix} p & q & r \\ a & b & c \\ -15 & -9 & -3 \end{vmatrix}$

e) $\begin{vmatrix} p+20 & q+12 & r+4 \\ -10 & -6 & -2 \\ a & b & c \end{vmatrix}$

f) $\begin{vmatrix} a & b & 0 \\ 5 & 3 & 0 \\ p & q & 0 \end{vmatrix}$

Matrix Algebra

***Equal Matrices*:** $A = B$ if A and B have the same dimensions and each entry a_{ij} in A is equal to the corresponding entry b_{ij} in B.

***Sums and Differences of Matrices*:** Only matrices of the same dimensions can be added or subtracted. To find a sum or difference, add or subtract corresponding entries of the two matrices.

***Commutative Property of Matrix Addition*:** $A+B=B+A$
***Associative Property of Matrix Addition*:** $(A+B)+C = A+(B+C)$

***Zero Matrix*:** A matrix whose entries are all equal to 0.

***Additive Identity Property of Matrix Addition*:** $A+0=0+A=A$

Scalar Multiplication*:** If k is a real number and A is a matrix, the matrix kA is the matrix formed by multiplying each entry in A by k. The number k is called a ***scalar*,** and the matrix $\boldsymbol{kA}$ is called a ***scalar multiple of A.

***Properties of Scalar Multiplication*:**
$k(hA) = (kh)A$
$(k+h)A = kA + hA$
$k(A+B) = kA + kB$

Examples: Let $A = \begin{bmatrix} -7 & 2 & 4 \\ 3 & -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 & -8 \\ 0 & 2 & -3 \end{bmatrix}$. Calculate:

a) $A+B$ b) $A-B$ c) $4A$ d) $2A-3B$

***Row Vector*:** A 1 by n matrix $R = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$

***Column Vector*:** An n by 1 matrix $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

***Product of a Row and a Column*:** $RC = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n$

Example: Find $\begin{bmatrix} 5 & 1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 1 \\ 7 \end{bmatrix}$

Matrix Multiplication: To multiply A and B, the number of columns of A must equal the number of rows of B. The product matrix will have the same number of rows as A and the same number of columns of B.

$$\begin{array}{ccc} A & & B \\ m \text{ by } r & & r \text{ by } n \end{array}$$

Must be the same.

AB is m by n.

The entry in row i, column j of the product matrix is the product of row i of A and column j of B. For example, to find the entry in row 2, column 3 of AB, multiply row 2 of A by column 3 of B.

Examples: Find the following products.

a) $\begin{bmatrix} 2 & 3 & -4 \\ 4 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -5 & 3 \\ 1 & 0 \\ 6 & 2 \end{bmatrix}$

b) $\begin{bmatrix} -5 & 3 \\ 1 & 0 \\ 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & -4 \\ 4 & 5 & 1 \end{bmatrix}$

★ **Matrix Multiplication Is Not Commutative!** $AB \neq BA$

***Associative Property of Matrix Multiplication*:** $A(BC) = (AB)C$

***Distributive Property*:** $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$

***Identity Matrix*:** The n by n identity matrix I_n is the n by n matrix with 1's along the main diagonal and 0's everywhere else.

eg) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

***Identity Property*:** If A is an m by n matrix, then $I_m A = A$ and $AI_n = A$.
If A is an n by n square matrix, $AI_n = I_n A = A$.

***Inverse of a Matrix*:** Let A be a square n by n matrix. If there exists an n by n matrix A^{-1}, read "A inverse," for which $AA^{-1} = A^{-1}A = I_n$, then A^{-1} is called the **inverse** of the matrix A.

***Nonsingular Matrix*:** A matrix that has an inverse. ***Singular Matrix*:** A matrix with no inverse.

Finding the Inverse of a Nonsingular Matrix

1. Form the matrix $[A \mid I_n]$.
2. Transform the matrix $[A \mid I_n]$ into reduced row echelon form.
3. The reduced row echelon form of $[A \mid I_n]$ will contain the identity matrix I_n on the left of the vertical bar; the n by n matrix on the right of the vertical bar is the inverse of A.

Examples: Find the inverse of each nonsingular matrix.

a) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Solving a System of Linear Equations Using an Inverse Matrix

The system of equations $\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$ can be rewritten as the matrix equation $AX = B,$ where

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Using matrix algebra, we can show:

$$\begin{aligned} &AX = B \\ &A^{-1}(AX) = A^{-1}B \\ &(A^{-1}A)X = A^{-1}B \\ &I_n X = A^{-1}B \\ &\boldsymbol{X = A^{-1}B} \end{aligned}$$

★ $\boldsymbol{X = A^{-1}B}$ means that to find the solution of the system of equations, multiply the inverse of the coefficient matrix by the column matrix formed from the constants on the right hand side of the equations.

Examples: Solve each system of equations using the inverses found in the previous example.

a) $\begin{cases} 5x - 2y = 3 \\ 3x - y = -4 \end{cases}$

b) $\begin{cases} x + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y = 6 \end{cases}$

Partial Fraction Decomposition

***Proper Fraction*:** Degree of numerator < Degree of denominator.
***Improper Fraction*:** Degree of numerator ≥ Degree of denominator.

Use long division to rewrite improper fractions as the sum of a polynomial and a proper rational expression.

Example: Tell whether the given rational expression is proper or improper. If improper, rewrite it as the sum of a polynomial and a proper rational expression.

a) $\dfrac{5x^3+2x-1}{x^2-4}$

b) $\dfrac{x}{x^2-1}$

Partial Fraction Decomposition:

Consider adding two rational expressions:

$$\frac{3}{x+4}+\frac{2}{x-3}=\frac{3(x-3)+2(x+4)}{(x+4)(x-3)}=\frac{3x-9+2x+8}{(x+4)(x-3)}=\frac{5x-1}{x^2+x-12}$$

In this section, we are going to reverse the procedure starting with the rational expression $\dfrac{5x-1}{x^2+x-12}$ and writing it as a sum of two simpler fractions $\dfrac{3}{x+4}$ and $\dfrac{2}{x-3}$.

This process is called ***partial fraction decomposition***, and the two simper fractions are called ***partial fractions*.**

Case 1: Denominator Has Only Nonrepeated Linear Factors

Given a rational expression with $Q(x)=(x-a_1)(x-a_2)...(x-a_n)$, the partial fraction decomposition of $\dfrac{P}{Q}$ takes the form $\dfrac{P(x)}{Q(x)}=\dfrac{A_1}{x-a_1}+\dfrac{A_2}{x-a_2}+...+\dfrac{A_n}{x-a_n}$, where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

To Solve:

1. Factor the denominator completely.
2. Set fraction equal to the sum of multiple fractions--one with each factor as a denominator. For numerators, use A, B, C etc. **Make sure you have written this step as an equation!**
3. Clear the equation of fractions by multiplying both sides of the equation by $Q(x)$.

Heaviside Method (Named after Oliver Heaviside, an English electrical engineer, mathematician, and physicist)**:**

4. You now have an equation that is true for any value of x. Choose values to plug in for x that will result in A, B, C, etc. being multiplied by zero and cancelling out. Determine as many of the numerators as possible in this way.
5. If you have not yet solved for all of the variables, plug in any other number(s) for x until you have an equation or a system of equations that can be solved for the remaining variables.
6. Substitute the solutions for A, B, etc. back into the partial fractions.

Coefficients Method:

1-3. Same as above.

4. Equate the coefficients of like powers and write a system of equations.
5. Solve the system for A, B, etc.
6. Substitute the solutions for A, B, etc. back into the partial fractions.

Examples: Write the partial fraction decomposition of the following.

a) $\dfrac{x+2}{x^2-7x+12}$

b) $\dfrac{x^2-9x-6}{x^3+x^2-6x}$

Case 2: Denominator Has Repeated Linear Factors

If the denominator has a repeated linear factor, $(x-a)^n$, where $n \geq 2$, then in the partial fraction decomposition, we allow for the terms $\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}$, where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

Examples: Write the partial fraction decomposition of the following.

a) $\frac{2x^2+7x+23}{(x-1)(x+3)^2}$

b) $\frac{x+1}{x^2(x-2)^2}$

Case 3: Denominator Has a Nonrepeated Irreducible Quadratic Factor

If the denominator contains a quadratic factor of the form ax^2+bx+c that can't be factored, then in the partial fraction decomposition, allow for the term $\dfrac{Ax+B}{ax^2+bx+c}$, where the numbers A and B are to be determined.

Examples: Write the partial fraction decomposition of the following.

a) $\dfrac{2x+4}{x^3-1}$

b) $\dfrac{7x^2-25x+6}{\left(x^2-2x-1\right)\left(3x-2\right)}$